# Convex Optimization Problem II

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**Things to know**

- Convex set and examples
- Visualization of a halfspace
- Convex preserving map

**Convex set** A subset $\mathcal{C}$ of $\mathbb{R}^n$ is convex if and only if for all $x_1, x_2 \in \mathcal{C}$ and all $\theta \in [0, 1]$,

$$\theta x_1 + (1 - \theta) x_2 \in \mathcal{C}$$

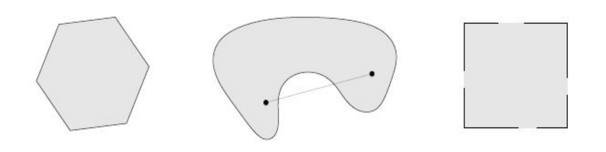


**Figure 1:** One convex set and two nonconvex sets

Cone: A set $K \subset \mathbb{R}^n$ is a cone if $x \in K \to \alpha x \in K$ for any $\alpha \geq 0 \in \mathbb{R}$.

Convex cone $C$ is convex and cone, which means that

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$.

Convex cone is convex.

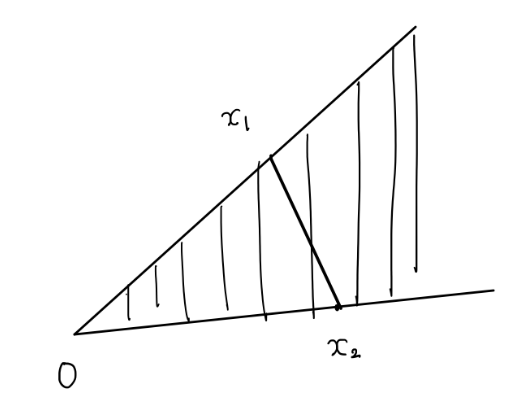



**Figure 2:** Convex cone

Norm cone

$$\mathcal{C} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ : \|x\|_2 \le t\}$$

Note that

$$C = \{(x,t) \in \mathbb{R}^{n+1} : \begin{pmatrix} x \\ t \end{pmatrix}^\top \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \le 0, t \ge 0\}$$

Norm cone is convex.

## **Example** (norm cone)

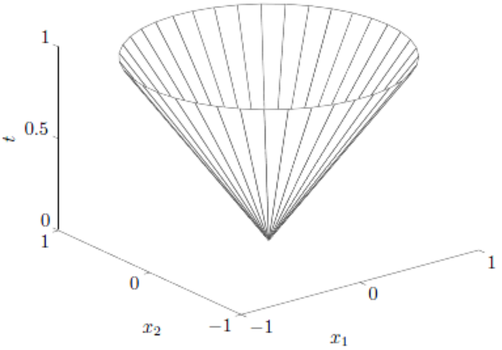
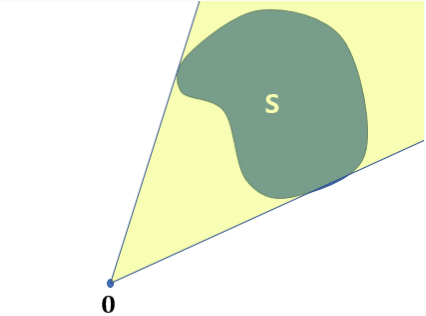



**Figure 2.10** Boundary of second-order cone in $\mathbf{R}^3$, $\{(x_1, x_2, t) \mid (x_1^2 + x_2^2)^{1/2} \leq t\}$.

Conic hull: the conic hull of a set $S$ is the set of all conic combinations of the points in S,

$$\text{Cone}(S) = \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \geq 0, x_i \in S \right\}.$$



(Conic hull) The conic hull of a set $C$ is the smallest convex cone containing $C$.

(proof)

- Conic hull of a set $C$ is convex and cone.
- If $C'$ is convex cone containing $C$ then the conic hull of $C$ is contained by $C'$.
- Thus, the proof is complete.

Ray: $\{x_0 + \theta\nu : \theta \geq 0\}$

A ray is convex.

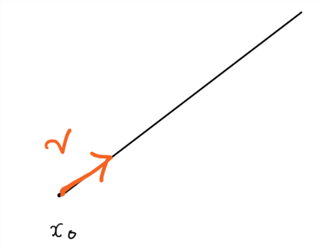



**Figure 3:** Visualization of Ray

## Hyperplane and Halfsapce

- Hyperplane: $\{x \in \mathbb{R}^n : a^\top x = b\}$
- Halfspace: $\{x \in \mathbb{R}^n : a^\top x \leq b\}$

Both hyperplane and halfspace are convex. Hyperplain divides the space into the two halfspace. Each halfspace is the SOLUTION SET OF THE INEQUALITY.

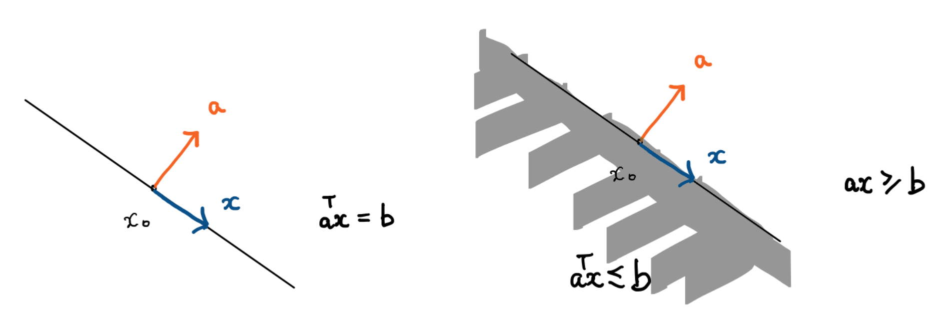
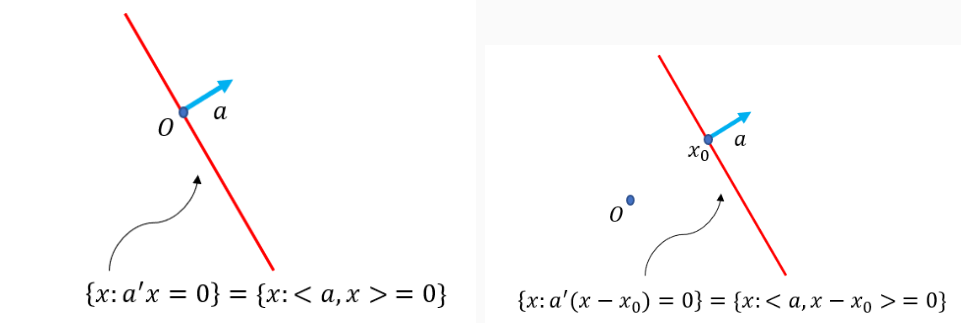



**Figure 4:** Hyperplane and Halfspace

## Visualization of half space

We know that $C = \{x : a^\top x = 0\}$ is illustrated by line on a plane and the vector can be denoted by normal vector at the origin of the line. Let $x = a$ then $a^\top x > 0$ and we know that the $x = a \in \{x : a^\top x \geq 0\}$. That is, the half space containing the denoted $a$ becomes $\{x : a^\top x \geq 0\}$.

Let $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Fix an $a$ and consider a set $\{x : a^\top x = 0\}$. The figure below shows how to display $\{x \in R^n : a^\top x = b\}$ on graph. First consider a set $\{x \in R^n : a^\top x = 0\}$.




Unless $a \neq 0$, we can choose $x_0 \in \mathbb{R}^n$ satisfying $a^\top x_0 = b$. Similarly, we can denote $a$ as a perpendicular vector on the line.

($\{x : a^\top x \geq b\}$ 영역의 시각화 방법)

- $\{x : a^\top x = b\}$ 을 나타내는 직선을 그린다.
- 직선의 위쪽의 한 점 $x_0$를 선택한 후 $a^\top x_0$를 계산한다.
- $a^\top x_0 \geq 0$ 가 참이면 해당 영역을 칠하고, 거짓이면 반대쪽 영역을 칠한다.

## Practice 1

Consider $a, x \in \mathbb{R}^2$

- Let $a = (2,1)^\top$ and draw the point $a$ in $\mathbb{R}^2$.
- Draw the line, the set of points $(x_1, x_2)$ satisfying $2x_1 + x_2 = 0$.
- Check that the line is the set $C = \{x : a^\top x = 0\}$.
- Make a shade on the region $2x_1 + x_2 \geq 0$ and check that the region contains $a$.
- Finally check that $a$ is orthogonal to all elements in $C$.

## Practice 2

Consider $A = \mathbb{R}^{m\times 2}$ and $x \in \mathbb{R}^2$. Denote $a_j^\top \in \mathbb{R}^2$ for $j = 1, \cdots, m$ be the $j$th row vector of $A$. We will illustrate the region $\{x : Ax \preceq 0\}$.

- Note that $\{x : Ax \preceq 0\} = \cap_{j=1}^m \{x : a_j^\top x \leq 0\}$.
- Draw the region, the set of points $(x_1, x_2)$ satisfying $a_j^\top x \leq 0$ for each $j$.
- Find the intersection of the regions.

**Examples** ($l_2$-ball)

$\mathcal{C} = \{\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$

Euclidean norm: $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$.

$$\mathbf{x} \in \mathcal{C} \Leftrightarrow \|\mathbf{x}\|_2 \leq 1$$

For $\mathbf{x}, \mathbf{x}' \in \mathcal{C}$,

$$\begin{aligned} \|\theta\mathbf{x} + (1-\theta)\mathbf{x}'\|_2 &\leq \|\theta\mathbf{x}\|_2 + \|(1-\theta)\mathbf{x}'\|_2 \\ &= \theta\|\mathbf{x}\|_2 + (1-\theta)\|\mathbf{x}'\|_2 \leq 1 \end{aligned}$$

$\theta\mathbf{x} + (1-\theta)\mathbf{x}' \in \mathcal{C}$.

The first inequality holds because of triangular inequality.

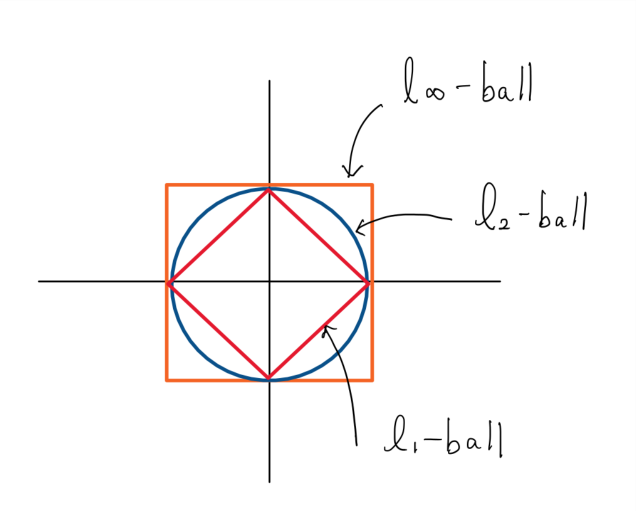



**Figure 5:** $l_p$-ball

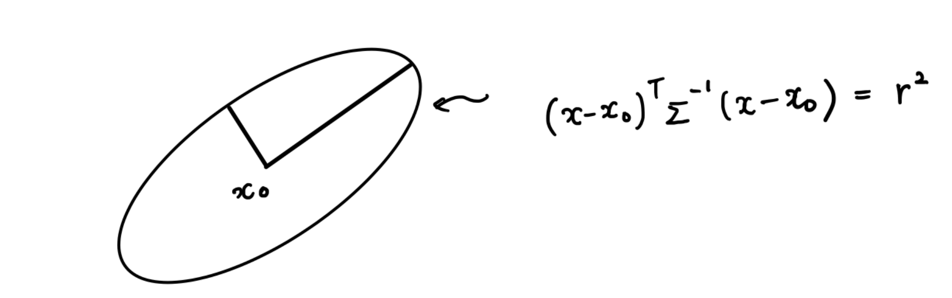



**Figure 6:** Ellipsoid

## **Examples** (polyhedron)

Let $A$ be $n \times p$ matrix, $b \in \mathbb{R}^p$, $\mathbf{C}$ be $m \times p$ matrix, and $d \in \mathbb{R}^m$.

$C = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$ is convex.

위 예는 Convex set의 정의를 이용해서 확인할 수 있는 예다.

(Proof) Let $x, x' \in C$ For $\theta \in [0, 1]$

$$A(\theta x + (1-\theta)x') = \theta Ax + (1-\theta)Ax' \leq \theta b + (1-\theta)b = b,$$

and

$$C(\theta x + (1-\theta)x') = \theta Cx + (1-\theta)Cx' = d$$

So, $\theta x + (1-\theta)x' \in C$

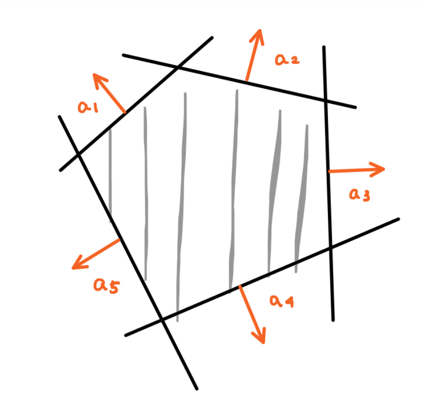



**Figure 7:** Visualization of Polyhedron

## supporting hyperplane

Let $C \subset \mathbb{R}^n$ and $x_0$ be a point on the boundary of $C$. If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$ then, the hyperplane $\{x : a^\top x = a^\top x_0\}$ is called a supporting hyperplane to $C$ at $x_0$.

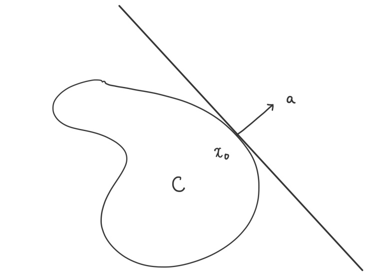



**Figure 8:** Caption

## Convex preserving operation

- If $C_1$, $C_2$ are convex sets, then $C_1 \cap C_2$ is also convex set.
- If $C$ is convex set, then $C + z = \{x + z : x \in C\}$ is convex.
- Let $C$ be convex subset of $\mathbb{R}^m$ and $f : x \in C \mapsto Ax + b \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. Then $\{y : y = Ax + b, x \in C\}$ is convex.
- Perspective function*
- Linear fractional functions*

## Perspective function

$f : \mathbb{R}^n \times \mathbb{R}_{++} \mapsto f(x, z) \in \mathbb{R}^n$, where $f(x, z) = x/z$.

The perspective function preserves the convexity of a set. If $C \subset \text{dom}(f)$ is convex, then $f(C)$ is convex. In addition, an inverse image of $f$ is convex.

## Linear fractional function

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. $f : \mathbb{R}^n \mapsto f(x) \in \mathbb{R}^m$, where

$$f(x) = \frac{Ax + b}{c^\top x + d}, \quad \mathsf{dom}(f) = \{x : c^\top x + d > 0\}.$$

Linear fractional function Let $g : \mathbb{R}^n \mapsto \mathbb{R}^{m+1}$ is affine, i.e.,

$$g(x) = \begin{pmatrix} A \\ c^\top \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Let $P$ be a perspective function, then

$$f(x) = P \circ g.$$

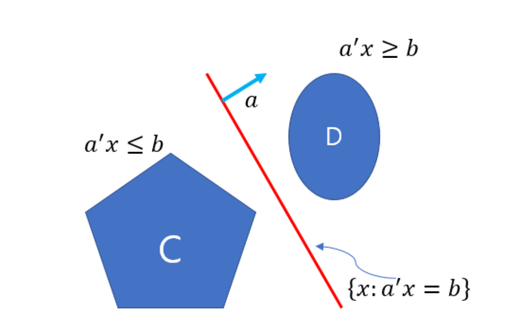
This representation of $f$ helps to prove that the linear-fractional function is convex preserving.

# HW

- Prove that the linear fractional function is convexity preserving map.
- Solve 2.1-2.10, 2.16-2.19

## Separating hyperplane theorem

Let $C$ and $D$ be convex sets in $\mathbb{R}^n$ with $C \cap D = \phi$. Then, there exists $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$, such that $a'x \leq b$ for all $x \in C$ and $a'x \geq b$ for all $x \in D$.



Strict separating hyperplane theorem

In general, a strict separating hyperplane does not hold, even when $C$ and $D$ are closed.

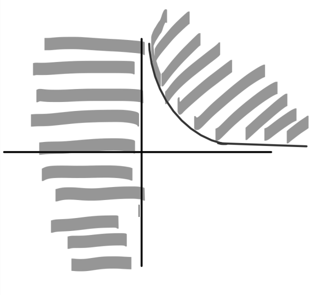


**Figure 9:** Not strict separating convex sets

# Appendix

Strict separating hyperplane theorem

Let $C$ be a closed convex set and $D = \{x_0\}$ with $x_0 \notin C$. Then, there exists $a \neq 0$ and $b$ such that $a^\top x < b$ for all $x \in C$ and $a^\top x > b$ for $x \in D$.

This result implies that a convex is represented by all intersections of hyperplanes containing the convex set.

# Appendix

Converse separating hyperplane theorem

See p50.